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Citation for published version:

Ducceschi, M & Bilbao, S 2016, 'Linear string vibrations in musical acoustics: Assessment and comparison of models', *The Journal of the Acoustical Society of America*, vol. 140, no. 4, pp. 2445-2456.
<https://doi.org/10.1121/1.4962553>

Digital Object Identifier (DOI):

[10.1121/1.4962553](https://doi.org/10.1121/1.4962553)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

The Journal of the Acoustical Society of America

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Linear stiff string vibrations in musical acoustics: assessment and comparison of models

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(Dated: October 17, 2015)

Strings are amongst the most common elements found in musical instruments and an appropriate physical description of their dynamics is essential to modelling, analysis and simulation. An ideal model should be able to describe the fundamental aspects of the physics of strings, as well as avoiding unnecessary complexities. Because strings are thick, stiffness must be taken into account in any suitable model. This paper presents and assesses three such models: Timoshenko, shear and Euler-Bernoulli.

PACS numbers:

I. INTRODUCTION

The simplest model of linear transverse string vibration is almost certainly the 1D wave equation; normally it is accompanied by additional terms modeling various effects, the most important of which is stiffness, the subject of this paper. Stiffness in strings leads to a progressive stretching or inharmonicity of the partials in the resulting sound, and is essential in any refined model of string vibration, as it leads to perceptually salient effects such as octave stretching, as well as to the reduction of beating phenomena when various notes are played simultaneously.

The most widely used stiff string model is a variant of the 1D wave equation incorporating a stiffness term as per the Euler-Bernoulli model of beam vibration. Here and henceforth in this article, such a model will be referred to as a stiff string of Euler Bernoulli type. Such an equation has been employed in a number of studies, especially in the case of finite difference simulations of piano strings. Notable works include those of Ruiz and Hiller and Ruiz, Bacon and Bowsher, Boutillon, Chaigne and Askenfelt⁸, and Giordano¹⁵. In the sound synthesis setting, such an equation has also been used as a starting point for digital waveguide models^{2,11}

The Euler-Bernoulli stiff string model is notable for its simplicity. It is known, however, that for such equation, phase and group velocity are unbounded in the limit of high frequency or wavenumber¹⁶, and it has been noted by some authors that this behaviour is unphysical. To address this shortcoming, more recent work has employed a more refined stiff string model, based on the Timoshenko theory of beams⁵⁻⁷. The Timoshenko theory can be written as a system of two coupled partial differential equations (PDEs) of second order, which can be combined into a single equation of fourth order in both space and time. The Timoshenko system is hyperbolic, and predicts finite group velocities, and thus avoids the artifacts of Euler-Bernoulli; in the low frequency range, how-

ever, the two models converge. The related issue of how large the differences between these two models are, and at which point in the spectrum they come into play, has not yet been addressed in the literature in musical acoustics, and this purpose of this article is to analyse and quantify such differences with regard to typical strings as they occur in musical instruments. Timoshenko and Euler-Bernoulli are, of course, only two among a large number of possible models. A third system will be considered here, known as the shear model. All such systems have been shown to be fairly good approximations to the exact 3D dynamics¹⁶.

The structure of this article is as follows. Section II presents the derivation of the Timoshenko model through a variational approach, and two simplified systems will be derived from it: shear and Euler-Bernoulli. For all models, boundary conditions, dispersion properties and modal frequencies will be derived. Section III draws a comparative analysis of the three systems, based on the results derived in the previous section. Finally, section IV presents a discussion based on the analysis and concluding remarks.

II. MODELS

A stiff string is modelled as a beam carrying tension along the longitudinal direction: physically, a stiff string is a "prestressed" rod, with appropriate stiffness and tension parameters. In this respect, the stiff string models discussed in this paper must draw from appropriate beam theories. Before proceeding, it is worth introducing a few strings of interest in musical acoustics. These are, in fact, amongst the thickest strings that one may encounter in instruments: double-bass $E1$; piano $D\#1$; acoustic guitar $E2$. In the remainder of the paper, they will be denoted as, respectively E_1^b ; $D_{\#1}^p$; E_2^g . All the strings are made of steel and have circular cross section. Note that winding is not considered for the strings in this paper. In fact, winding is a technique that allows to increase the mass of a string by covering the steel core with a denser metal (usually copper) without changing considerably its stiffness (and thus inharmonicity). As Fletcher

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points out¹²: "the elastic-restoring torque is due almost entirely to the steel core, but the linear density is due to the core and the windings". The string parameters are summarised in table I. Note that the area A and the

STRING PARAMETERS

	r (mm)	L_0 (mm)	T_0 (N)
E_1^b	1.50	1.10	450
$D_{\#1}^p$	0.74	1.94	310
E_2^g	0.71	0.67	150

TABLE I. Case studies: double-bass low E E_1^b ; piano low D# $D_{\#1}^p$ and acoustic guitar low E E_2^g . All strings are made of steel, with $\rho = 7860 \text{ kg/m}^3$, $E = 2.02 \cdot 10^{11} \text{ Pa}$, $G = 7.77 \cdot 10^{10} \text{ Pa}$ and $\kappa = 0.89$.

area moment of inertia I are readily calculated from the radius r as

$$A = \pi r^2; \quad I = Ar^2/4\pi.$$

Choosing an appropriate beam model is essential as the base for stiff strings. Exact 3D models have been derived for finite-element application, see for example the works by Jelenić and Crisfield²⁰, Betsch and Steinmann³ and the book by Kolsky²². Approximate (or engineering) models can be derived by averaging out the effects on the plane perpendicular to transverse motion, and four such models are prominent in the literature: Timoshenko, shear, Rayleigh and Euler-Bernoulli^{16,17}. The Timoshenko model is widely considered to be the best approximation to the exact models, and, in fact, the other three models can be viewed as further simplifications to the Timoshenko system. Thus, the presentation will start from this system, which will then be simplified to yield the other subsystems.

A. Timoshenko

In the absence of prestress, the Timoshenko system reads¹⁶

$$\begin{aligned} \rho A w_{,tt} &= A\kappa G w_{,xx} - A\kappa G \phi_{,x} \\ \rho I \phi_{,tt} &= EI \phi_{,xx} + A\kappa G w_{,x} - A\kappa G \phi \end{aligned}$$

In the equations, G is the shear modulus, κ is a correction factor known as Timoshenko shear coefficient, I is the area moment of inertia, E is Young's modulus and ϕ is the angle which measures the rotation of the cross section during bending with respect to the section normal to the neutral line. Note that, for isotropic materials, $G = E/2(1 + \nu)$, where ν is Poisson's ratio ($\nu = 0.3$ for steel); and for circular cross section $\kappa = 6(1 + \nu)/(7 + 6\nu)$ ¹⁷. The system can be scaled to yield a form with less constant parameters. To this extent, consider the following nondimensional variables denoted by overbars

$$\bar{w} = w/w_0; \quad \bar{\phi} = \phi/\phi_0; \quad \bar{x} = x/X; \quad \bar{t} = t/T;$$

and the following relations

$$X^2 = \frac{I}{A}; \quad T^2 = \frac{\rho I}{A\kappa G}; \quad \phi_0 = \frac{w_0}{X}; \quad \beta = \frac{E}{\kappa G}.$$

With this in mind, and avoiding overbars, the Timoshenko system reads

$$w_{,tt} = w_{,xx} - \phi_{,x} \quad (1a)$$

$$\phi_{,tt} = \beta \phi_{,xx} + w_{,x} - \phi \quad (1b)$$

The system may be arrived at by means of different techniques. One may choose to draw a free body diagram and balance moments and forces; alternatively (and this is the approach shown here) one may derive the kinetic and potential energies from elasticity theory considerations, and perform calculus of variations. If a strong solution to this system exists (and it does, as proven by Chabassier *et.al*⁷), over the domain $x \in [0, L] \triangleq \mathcal{D}$ then w, ϕ and their derivatives up to the order 2 necessarily belong to a set $V : \mathcal{D} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $v(x, t) \in V \subseteq \mathcal{C}^0(L^2(\mathcal{D}); \mathbb{R}^+)$. It is then possible to define, for two functions $v_1, v_2 \in V$, the following scalar product and norm

$$\langle v_1, v_2 \rangle_{\mathcal{D}} = \int_0^L v_1 v_2 dx; \quad \|v_1\|_{\mathcal{D}}^2 = \langle v_1, v_1 \rangle_{\mathcal{D}}.$$

With this in mind, the Hamiltonian \mathcal{H} of the system reads¹⁷

$$\mathcal{H} = \underbrace{\frac{\|w_{,t}\|_{\mathcal{D}}^2}{2}}_{\mathcal{K}_b} + \underbrace{\frac{\|\phi_{,t}\|_{\mathcal{D}}^2}{2}}_{\mathcal{K}_s} + \underbrace{\frac{\beta \|\phi_{,x}\|_{\mathcal{D}}^2}{2}}_{\mathcal{U}_b} + \underbrace{\frac{\|\phi - w_{,x}\|_{\mathcal{D}}^2}{2}}_{\mathcal{U}_s}$$

It is composed of kinetic (\mathcal{K}) and potential (\mathcal{U}) energy terms, where the subscripts b, s stand for "bending" and "shear" respectively. The Lagrangian \mathcal{L} is obtained from here as

$$\mathcal{L} = \mathcal{K} - \mathcal{U}$$

System (1) can be derived by considering the variation of such Lagrangian by means of two admissible functions $\tilde{w} = w + \delta w$; $\tilde{\phi} = \phi + \delta \phi$ and two instants in time t_0, t_1 such that $\delta w(t_0) = \delta w(t_1) = \delta \phi(t_0) = \delta \phi(t_1) = 0$. Calculating the minimum of the functional

$$F[\tilde{w}, \tilde{\phi}] \triangleq \int_{t_0}^{t_1} \mathcal{L}(\tilde{w}, \tilde{\phi}) dt$$

leads to system (1) with the associated boundary conditions¹⁴. The case of prestressed Timoshenko beam can be treated in an analogous way, provided that one is able to define an added potential energy for tension. This is an important point, because the literature presents at least two different versions for this added energy²⁵. Assuming the prestress to be T_0 (uniform along the length of the beam) in the scaled form these are two forms are

$$\mathcal{U}_t^1 = \frac{\alpha - 1}{2} \left[\langle w_{,x}, \phi \rangle_{\mathcal{D}} - \|\phi\|_{\mathcal{D}}^2 \right]; \quad \mathcal{U}_t^2 = \frac{\alpha - 1}{2} \|w_{,x}\|_{\mathcal{D}}^2;$$

where the subscript t stands for tension and

$$\alpha = 1 + \frac{T_0}{A\kappa G}. \quad (2)$$

Inserting either form into the Lagrangian and calculating the variations leads to two different prestressed Timoshenko systems. In terms of the parameters ϵ_1, ϵ_2 , model 1 is obtained by considering $\epsilon_1 = 1; \epsilon_2 = 0$; model 2 is obtained by instead using $\epsilon_1 = 0; \epsilon_2 = 1$. The system reads

$$w_{,tt} = [1 + \epsilon_2(\alpha - 1)]w_{,xx} - [1 - \epsilon_1(\alpha - 1)]\phi_{,x} \quad (3a)$$

$$\phi_{,tt} = \beta\phi_{,xx} + [1 - \epsilon_1(\alpha - 1)](w_{,x} - \phi) \quad (3b)$$

A literary survey reveals that the choice of one model

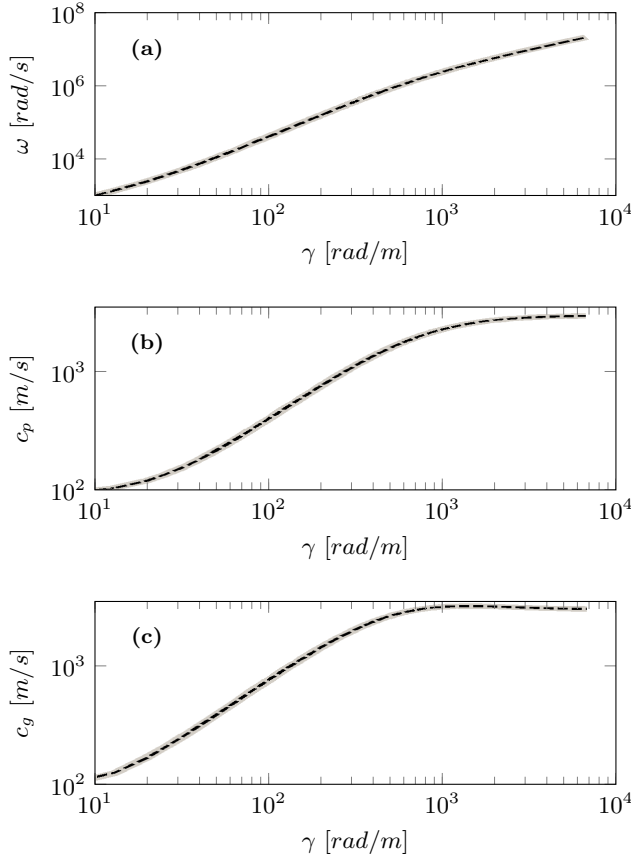


FIG. 1. The double-bass E_1^b string. Comparison of prestressed Timoshenko models, as per Eq.(3). For all figures, thick grey line is model 1, dotted black line is model 2. (a): dispersion relation. (b): phase velocity. (c): group velocity.

over the other is still matter of debate: Kounadis²³ derived model 1 from a free body diagram, but it was later shown by Sato²⁵ that that both models can actually be written in terms of Hamilton's principle. Later, Djonjorov and Vassilev¹⁰ pointed out that the choice should be made according to which model is able to predict accurately the "critical load" after which the system undergoes dynamical instability (buckling): a series of papers address this issue but the question remains open. For musical instruments, however, it must be pointed out

that typical tensions for are way far from being "critical": typically, the ratio $T_0/A\kappa G$ approaches 10^{-3} or smaller. It is interesting to plot the dispersion relation, phase and group velocities for the thick E_1^b string of table I, for both model 1 and model 2. This is done in figure 1 where the two models appear to overlap completely over a vast range of wavenumbers. This is not surprising, given the low tension value with respect to shear force. In turn, the two models in the context of musical acoustics are expected to produce the same results. Model 2 was recently used by Chabassier *et.al*⁵ to model and simulate the grand-piano and such model will be employed here too. Hence, for the remainder of the paper the scaled, prestressed Timoshenko system (TM) reads

$$TM : \quad w_{,tt} = \alpha w_{,xx} - \phi_{,x} \quad (4a)$$

$$\phi_{,tt} = \beta \phi_{,xx} + w_{,x} - \phi \quad (4b)$$

1. Boundary Conditions

Boundary conditions for the prestressed Timoshenko system may be obtained by varying the Lagrangian. Classic simply-supported, clamped and free conditions are obtained as

$$\begin{aligned} SS : \quad w = \phi_{,x} = 0; \quad CL : \quad w = \phi = 0; \\ FF : \quad \alpha w_{,x} - \phi = \phi_{,x} = 0. \end{aligned} \quad (5)$$

The simply-supported conditions describe a fixed edge with vanishing moment (ϕ_x). For the clamped case, the edge and the cross section are fixed (therefore these conditions are purely geometrical); for the free case both the moment and the shear force vanish. The same boundary conditions may be re-stated in terms of the function w only, by considering Eqs.(4a) and (4b). The conditions are summarised in table II. Such form of the boundary conditions is of course less compact, but nonetheless useful when the modes for the transverse displacement w are sought. This will be accomplished in section II.A.3.

2. Dispersion Relations

Systems in which different wavelengths travel at different speeds are called dispersive. Dispersion is directly related to inharmonicity, as standing waves present wavelengths that are not multiples of each other. Dispersion relation for waves are obtained by injecting a plane wave in the system and observing the relation between the wave number and the frequency of vibration. Hence, a plane wave of the form

$$\begin{pmatrix} w \\ \phi \end{pmatrix} = \begin{pmatrix} d_w \\ d_\phi \end{pmatrix} e^{j(\omega t - \gamma x)}$$

is injected into (4), to get

$$\begin{pmatrix} -\omega^2 + \alpha\gamma^2 & -j\gamma \\ j\gamma & -\omega^2 + \beta\gamma^2 + 1 \end{pmatrix} \begin{pmatrix} d_w \\ d_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

BOUNDARY CONDITIONS

	SS	CL	FF
TM	w	w	$-(\alpha + \beta)w_{,ttx} + (1 - \alpha)w_{,x} + \alpha\beta w_{,xxx}$
	$w_{,xx}$	$\alpha\beta w_{,xxx} + w_{,x} - \beta w_{,xtt}$	$w_{,tt} - \alpha w_{,xx}$
SH	w	w	$-\beta w_{,ttx} + (1 - \alpha)w_{,x} + \alpha\beta w_{,xxx}$
	$w_{,xx}$	$\alpha\beta w_{,xxx} + w_{,x} - \beta w_{,xtt}$	$w_{,tt} - \alpha w_{,xx}$
EB	w	w	$(1 - \alpha)w_{,x} + \beta w_{,xxx}$
	$w_{,xx}$	$w_{,x}$	$w_{,xx}$

TABLE II. Summary of classical boundary conditions for the Timoshenko (TM), shear (SH) and Euler-Bernoulli (EB) models. All quantities in the table vanish at the boundary.

In order for this system to have nontrivial solutions, the determinant of the matrix must be zero. This gives

$$\omega^4 + (-\alpha\gamma^2 - \beta\gamma^2 - 1)\omega^2 + \alpha\gamma^2 - \gamma^2 + \alpha\beta\gamma^4 = 0.$$

This is a fourth order equation in ω which is solved by

$$\begin{aligned} \omega_{\pm}^2 &= \frac{(\alpha + \beta)\gamma^2 + 1}{2} \\ &\pm \frac{[(\beta - \alpha)^2\gamma^4 + 2(\beta - \alpha + 2)\gamma^2 + 1]^{1/2}}{2} \\ &\triangleq P_1(\gamma^2) \pm [P_2(\gamma^2)]^{1/2} \end{aligned}$$

Note that the Timoshenko systems leads to *two* different dispersion curves: one curve (denoted by $+$) is related to the shear mode, the other curve ($-$) is the flexural waves mode. The square root in the formula for ω^2 can potentially be an imaginary number. In fact, the radicand is a parabola in γ^2 , which is 1 for $\gamma^2 = 0$ and which has positive curvature and whose minimum is 1. Hence, $\omega^2 \in \mathbb{R} \ \forall \gamma^2 \in \mathbb{R}^+$. The sign of ω^2 must as well be discussed. Now, the sign of ω_+^2 is surely positive. For ω_-^2 the sign must be checked; ω_-^2 is in the form of $P_1(\gamma^2) - [P_2(\gamma^2)]^{1/2}$ where $P_1(0) = P_2(0) = 1$ and both are monotonically increasing functions. In the limit of $\gamma \rightarrow \infty$, $P_1 > [P_2(\gamma^2)]^{1/2}$, therefore, because of monotonicity, $\omega_-^2 < 0 \Leftrightarrow P_1 = P_2^{1/2}$ for some real positive values of γ^2 . However, the equation $P_1 = P_2^{1/2}$ gives two negative solutions ($\gamma_{\pm}^2 = [-\beta - 1 \pm (2\beta + 1)^{1/2}]/\beta^2$) therefore $\omega_-^2 > 0 \ \forall \gamma^2 \in \mathbb{R}^+$. Now that the sign of ω_{\pm}^2 has been established, asymptotic solutions are sought. In the limit of small and large wavenumbers, one has

$$\lim_{\gamma \rightarrow 0} \omega_{\pm}^2 = \begin{cases} 1 + (\beta + 1)\gamma^2 \\ (\alpha - 1)\gamma^2 \end{cases} ; \quad \lim_{\gamma \rightarrow \infty} \omega_{\pm}^2 = \begin{cases} \beta\gamma^2 \\ \alpha\gamma^2 \end{cases}$$

These limits are important because they allow to get a glimpse on the behaviour of the phase and group velocities for the Timoshenko system. The two velocities are defined as

$$c_p = \frac{\omega}{\gamma}; \quad c_g = \frac{d\omega}{d\gamma}.$$

Starting with the phase velocity one gets

$$\lim_{\gamma \rightarrow 0} c_{p\pm} = \begin{cases} \infty \\ (\alpha - 1)^{1/2} \end{cases} ; \quad \lim_{\gamma \rightarrow \infty} c_{p\pm} = \begin{cases} \beta^{1/2} \\ \alpha^{1/2} \end{cases}$$

Therefore, the phase velocity of the shear mode is unbounded for small wavenumbers. However, both phase velocities are bounded at infinity. The group velocities are readily obtained as

$$\lim_{\gamma \rightarrow 0} c_{g\pm} = \begin{cases} (\beta + 1)^{1/2} \\ (\alpha - 1)^{1/2} \end{cases} ; \quad \lim_{\gamma \rightarrow \infty} c_{g\pm} = \begin{cases} \beta^{1/2} \\ \alpha^{1/2} \end{cases}$$

Hence the group velocity is always bounded. Note that, for $\alpha = 1$ (i.e. in the absence of tension T_0), the usual phase and group velocities for the Timoshenko beam are recovered¹⁶. Note as well that the flexural waves are non dispersive at both low and high frequencies (although with different speeds), whereas the shear waves are nondispersive at large frequencies.

3. Modes

When the system is bounded, the travelling wave description is not the most appropriate, as discrete modal shapes and frequencies appear in the system. For the Timoshenko model are defined in the following way

$$\begin{pmatrix} w \\ \phi \end{pmatrix} = e^{j\omega t} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} \quad (6)$$

In the equation, ω are the frequencies of vibration of the associated modal shapes Ψ , Φ . Substitution of (6) into (4) gives

$$-\omega^2 \Psi = \alpha \Psi_{,xx} - \Phi_{,x} \quad (7a)$$

$$-\omega^2 \Phi = \beta \Phi_{,xx} + \Psi_{,x} - \Phi \quad (7b)$$

System (7) is solved by considering a solution of the form

$$\begin{pmatrix} \Psi \\ \Phi \end{pmatrix} = \begin{pmatrix} d\Psi \\ d\Phi \end{pmatrix} e^{rx}$$

with $r \in \mathbb{C}$. When this solution is substituted into (7), one obtains

$$\begin{pmatrix} \omega^2 + \alpha r^2 & -r \\ r & \omega^2 - 1 + \beta r^2 \end{pmatrix} \begin{pmatrix} d_\Psi \\ d_\Phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The characteristic equation is a fourth-order polynomial in r

$$\alpha\beta r^4 + r^2[(\alpha + \beta)\omega^2 - \alpha + 1] + \omega^4 - \omega^2 = 0$$

with solutions

$$2\alpha\beta r_\pm^2 = \alpha - 1 - (\alpha + \beta)\omega^2 \pm \left[(\alpha - \beta)^2\omega^4 + 2(\alpha + \beta + \alpha\beta - \alpha^2)\omega^2 + (\alpha - 1)^2 \right]^{1/2} \triangleq P_1(\omega^2) \pm [P_2(\omega^2)]^{1/2}.$$

If $P_2(\omega^2) < 0$, then r_\pm^2 are complex numbers. However, $P_2(\omega^2)$ is a parabola in ω^2 , which has positive curvature and is positive for $\omega^2 = 0$. Therefore, if the slope at $\omega^2 = 0$ is positive, $P_2(\omega^2)$ remains positive $\forall \omega^2 \in \mathbb{R}^+$. The slope is positive $\leftrightarrow 2\alpha\beta - (\alpha - 1)(\alpha + \beta) > 0$. Solving for β gives $\alpha \frac{\alpha-1}{\alpha+1} < \beta$. Because $\alpha > 1$, note that $\alpha \frac{\alpha-1}{\alpha+1}$ is bounded from above by $\alpha - 1$, and because for the strings of interest here $\alpha - 1 < \beta$ (i.e. $EA > T_0$) then $P_2 > 0 \forall \omega^2 \in \mathbb{R}^+$. The sign of r_\pm^2 must as well be discussed. Such discussion is a little lengthy but not difficult, and the result is

$$r_+^2 = \begin{cases} > 0 & \text{for } \omega^2 < 1 \\ < 0 & \text{for } \omega^2 > 1 \end{cases} ; \quad r_-^2 = \begin{cases} < 0 & \text{for } \omega^2 < 1 \\ < 0 & \text{for } \omega^2 > 1 \end{cases}$$

The solutions for $\omega^2 > 1$ are not of interest here because they correspond to the high-frequency shear-wave mode. Transverse waves happen within the range $\omega^2 < 1$. Because of the sign of r_\pm^2 , a general solution for the modes is

$$\Psi = d_1 \sin(\lambda_- x) + d_2 \cos(\lambda_- x) + d_3 \sinh(\lambda_+ x) + d_4 \cosh(\lambda_+ x),$$

where $\lambda_+ = r_+$, $\lambda_- = \sqrt{|r_-^2|}$. This solution is then inserted into the selected boundary conditions (presented in table II), which give four equations of the form

$$\mathbf{A} \mathbf{d} = \mathbf{0}$$

where \mathbf{A} is a 4×4 matrix and $\mathbf{d} = {}^T(d_1, d_2, d_3, d_4)$. Nullifying the determinant of \mathbf{A} gives a frequency equation in the unknown ω^2 . Such equation is transcendental and must be solved numerically. A summary of such frequency equations for the classic combinations of boundary conditions is offered in table III.

B. Shear

The shear model is derived by neglecting the inertia of the cross section in Eq.(4b)¹⁷. In fact, the shear model may be arrived at by considering the asymptotic solution of Timoshenko for large wavelengths, as shown by

Hodges¹⁸. The validity of such assumption was later debated by Aristizabal-Ochoa¹, who points out that the shear beam with at least one free end and at most one rotationally constrained end invalidates the conservation of angular momentum, as proven by Kausel²¹. For musical acoustics, however, strings are fixed and therefore the shear model is a valid approximation to Timoshenko. Hence, the prestressed shear system (SH) reads

$$SH : \quad w_{,tt} = \alpha w_{,xx} - \phi_{,x} \quad (8a)$$

$$0 = \beta \phi_{,xx} + w_{,x} - \phi \quad (8b)$$

In this system, the shear force is still taken into account but the absence of rotational inertia forbids the development of shear waves. Hence, in this system only bending waves are present.

1. Boundary Conditions

The Lagrangian for the prestressed shear system is obtained from that of Timoshenko by neglecting the kinetic component of the cross section, \mathcal{K}_s . Calculus of variation allows to obtain the boundary conditions as follows

$$\begin{aligned} SS : \quad w = \phi_{,x} = 0; \quad CL : \quad w = \phi = 0; \\ FF : \quad \alpha w_{,x} - \phi = \phi_{,x} = 0. \end{aligned} \quad (9)$$

Note that these are formally identical to those for Timoshenko. The boundary conditions in the sole variable w are given in table II.

2. Dispersion Relation

A plane wave solution of the form

$$\begin{pmatrix} w \\ \phi \end{pmatrix} = \begin{pmatrix} d_w \\ d_\phi \end{pmatrix} e^{j(\omega t - \gamma x)}$$

is inserted in (8) to obtain

$$\begin{pmatrix} \omega^2 - \alpha\gamma^2 & j\gamma \\ -j\gamma & -\beta\gamma^2 - 1 \end{pmatrix} \begin{pmatrix} d_w \\ d_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Nullifying the determinant gives

$$\omega^2 = \frac{\gamma^2(\alpha\beta\gamma^2 + \alpha - 1)}{\beta\gamma^2 + 1}$$

Now, ω^2 is surely positive (from the definition of α in Eq.(2) one has immediately $\alpha > 1$). Again, asymptotic solutions are sought. in the small and large wavenumber limits

$$\lim_{\gamma \rightarrow 0} \omega^2 = (\alpha - 1)\gamma^2; \quad \lim_{\gamma \rightarrow \infty} \omega^2 = \alpha\gamma^2.$$

These limits yield the following phase and group velocities

$$\lim_{\gamma \rightarrow 0} [c_p, c_g] = (\alpha - 1)^{1/2}; \quad \lim_{\gamma \rightarrow \infty} [c_p, c_g] = \alpha^{1/2}$$

Like in the Timoshenko model, waves are nondispersive at low and high frequencies, with the same asymptotes.

SS:	$\sin(\lambda_- L) \sinh(\lambda_+ L) = 0$
CL:	$C_{cl} \sin(\lambda_- L) \sinh(\lambda_+ L) - \cos(\lambda_- L) \cosh(\lambda_+ L) + 1 = 0$
$C_{cl} =$	$\frac{\left[\beta^2(\lambda_-^2 - \lambda_+^2)\omega^4 + (\beta(\lambda_- - \lambda_+)(\beta(\lambda_+^3 - \lambda_-^3)\alpha + \lambda_- + \lambda_+) - \beta(\lambda_- + \lambda_+)(\beta(\lambda_-^3 + \lambda_+^3)\alpha + \lambda_+ - \lambda_-))\omega^2 - (\beta(\lambda_+^3 - \lambda_-^3)\alpha + \lambda_- + \lambda_+)(\beta(\lambda_-^3 + \lambda_+^3)\alpha + \lambda_+ - \lambda_-) \right]}{\left[(2\lambda_- \lambda_+ (\alpha\beta\lambda_+^2 + \beta\omega^2 + 1)(\alpha\beta\lambda_-^2 - \beta\omega^2 - 1)) \right]}$
FF:	$C_{ff} \sin(\lambda_- L) \sinh(\lambda_+ L) - \cos(\lambda_- L) \cosh(\lambda_+ L) + 1 = 0$
$C_{ff} =$	$\frac{\left[-(\alpha + \beta)(\lambda_- + \lambda_+)\omega^4 + (\alpha(\lambda_- + \lambda_+) - \lambda_+ - \lambda_- + \alpha\beta(\lambda_- + \lambda_+)^2(\lambda_- - \lambda_+) + \alpha^2\lambda_- \lambda_+(\lambda_- - \lambda_+))\omega^2 + \alpha\lambda_- \lambda_+(\lambda_- - \lambda_+ + \alpha(\lambda_+ - \lambda_- + \beta\lambda_- \lambda_+(\lambda_- + \lambda_+))) \right]}{\left[((2\lambda_- \lambda_+ (\alpha\lambda_-^2 - \omega^2)(\alpha\lambda_+^2 + \omega^2)(\alpha\omega^2 - \alpha + \beta\omega^2 - \alpha\beta\lambda_-^2 + 1)(\alpha\omega^2 - \alpha + \beta\omega^2 + \alpha\beta\lambda_+^2 + 1))) \right]}$

TABLE III. Summary of frequency equations for a prestressed Timoshenko beam, for the classical simply-supported, clamped and free end conditions.

3. Modes

The modes are defined in the same way as for the Timoshenko case

$$\begin{pmatrix} w \\ \phi \end{pmatrix} = e^{j\omega t} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} \quad (10)$$

Substitution of (10) into (8) gives

$$-\omega^2 \Psi = \alpha \Psi_{,xx} - \Phi_{,x} \quad (11a)$$

$$0 = \beta \Phi_{,xx} + \Psi_{,x} - \Phi \quad (11b)$$

The system is solved by considering a solution of the form

$$\begin{pmatrix} \Psi \\ \Phi \end{pmatrix} = \begin{pmatrix} d_\Psi \\ d_\Phi \end{pmatrix} e^{rx}$$

with $r \in \mathbb{C}$. When this solution is substituted into (11), one obtains

$$\begin{pmatrix} \omega^2 + \alpha r^2 & -r \\ r & -1 + \beta r^2 \end{pmatrix} \begin{pmatrix} d_\Psi \\ d_\Phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The characteristic equation is a fourth-order polynomial in r

$$\alpha\beta r^4 + r^2[\beta\omega^2 - \alpha + 1] - \omega^2 = 0$$

with solutions

$$2\alpha\beta r_\pm^2 = \alpha - 1 - \beta\omega^2 \pm \left[\beta^2\omega^4 + 2\beta(\alpha + 1)\omega^2 + (\alpha - 1)^2 \right]^{1/2}$$

In this case $r_+^2 > 0 \forall \omega^2 \in \mathbb{R}^+$ and $r_-^2 < 0 \forall \omega^2 \in \mathbb{R}^+$. Hence the modal shapes can be written as

$$\Psi = d_1 \sin(\lambda_- x) + d_2 \cos(\lambda_- x) + d_3 \sinh(\lambda_+ x) + d_4 \cosh(\lambda_+ x),$$

where $\lambda_+ = r_+$, $\lambda_- = \sqrt{|r_-^2|}$. Inserting this solution into the chosen boundary conditions in table II gives the following equation

$$\mathbf{A} \mathbf{d} = \mathbf{0}$$

where \mathbf{A} is a 4×4 matrix and $\mathbf{d} = {}^T(d_1, d_2, d_3, d_4)$. The associated transcendental frequency equations are given in table IV.

C. Euler-Bernoulli

The Euler-Bernoulli (EB) model is obtained from the shear model in the following way. Substitute $\phi = e^{j(\omega t - \gamma x)}$ in Eq.(8b) and obtain $\phi = w_{,x}/(\gamma^2\beta + 1)$. Expanding the denominator in the limit of small wavelengths γ gives $\phi \approx (1 - \gamma^2\beta)w_{,x}$ and by transforming

$$SS: \quad \sin(\lambda_- L) \sinh(\lambda_+ L) = 0$$

$$CL: \quad C_{cl} \sin(\lambda_- L) \sinh(\lambda_+ L) - \cos(\lambda_- L) \cosh(\lambda_+ L) + 1 = 0$$

$$C_{cl} = \frac{\left[\beta(\lambda_+^2 - \lambda_-^2)\omega^4 + 2\beta(\alpha\beta(\lambda_-^4 + \lambda_+^4) - \lambda_-^2 + \lambda_+^2)\omega^2 - \alpha\beta(\alpha\beta(\lambda_-^6 + \lambda_+^6) + 2\lambda_-^4 + 2\lambda_+^4) - \lambda_-^2 + \lambda_+^2 \right]}{\left[2\lambda_- \lambda_+ (-\alpha\beta\lambda_-^2 + \beta\omega^2 + 1)(\alpha\beta\lambda_+^2 + \beta\omega^2 + 1) \right]}$$

$$FF^*: \quad C_{ff} \sin(\lambda_- L) \sinh(\lambda_+ L) - \cos(\lambda_- L) \cosh(\lambda_+ L) + 1 = 0$$

$$C_{ff} = \frac{\left[(\beta(\lambda_- - \lambda_+)\omega^4 - (\lambda_- - \lambda_+)(\alpha\beta\lambda_-^2 - \alpha\beta\lambda_+^2 + \alpha - 1)\omega^2 - \alpha\lambda_- \lambda_+ (\alpha\lambda_- - \lambda_+ - \lambda_- + \alpha\lambda_+ - \alpha\beta\lambda_- \lambda_+^2 + \alpha\beta\lambda_-^2 \lambda_+)) \right. \\ \left. (-\beta(\lambda_- + \lambda_+)\omega^4 + (\lambda_- + \lambda_+)(\alpha\beta\lambda_-^2 - \alpha\beta\lambda_+^2 + \alpha - 1)\omega^2 + \alpha\lambda_- \lambda_+ (\lambda_- - \lambda_+ - \alpha\lambda_- + \alpha\lambda_+ + \alpha\beta\lambda_- \lambda_+^2 + \alpha\beta\lambda_-^2 \lambda_+)) \right]}{\left[2\lambda_- \lambda_+ (\alpha\lambda_-^2 - \omega^2)(\alpha\lambda_+^2 + \omega^2)(\alpha\beta\lambda_-^2 - \beta\omega^2 + \alpha - 1)(\alpha\beta\lambda_+^2 + \beta\omega^2 - \alpha + 1) \right]}$$

TABLE IV. Summary of frequency equations for a prestressed shear beam, for the classical simply-supported, clamped and free end conditions. * Note that the FF conditions violate the principle of conservation of angular momentum, as pointed out in¹. They are here reported for completeness.

back to the time domain one gets $\phi = w_{,x} - \beta w_{,xxx}$. This relation is then substituted into (8a) to obtain

$$EB: \quad w_{,tt} - (\alpha - 1) w_{,xx} + \beta w_{,xxxx} = 0. \quad (12)$$

1. Boundary Conditions

For Euler-Bernoulli the Hamiltonian is

$$\mathcal{H} = \underbrace{\frac{\|w_t\|_{\mathcal{D}}^2}{2}}_{\mathcal{K}_b} + \underbrace{\frac{(\alpha - 1)\|w_x\|_{\mathcal{D}}^2}{2}}_{\mathcal{U}_t} + \underbrace{\frac{\beta\|w_{,xx}\|_{\mathcal{D}}^2}{2}}_{\mathcal{U}_b}$$

and the boundary conditions are again obtainable through a variational approach. The result is

$$\begin{aligned} SS: \quad w = w_{,xx} = 0; \quad CL: \quad w = w_{,x} = 0; \\ FF: \quad w_{,xx} = (1 - \alpha)w_{,x} + \beta w_{,xxx} = 0. \end{aligned} \quad (13)$$

2. Dispersion Relation

For Euler-Bernoulli, a solution of the form $w = e^{j(\omega t - \gamma x)}$ is inserted in (12). This gives the following dispersion relation

$$\omega^2 = \gamma^2[(a + 1) + \gamma^2\beta].$$

In this case one sees immediately that

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \omega^2 &= (a - 1)\gamma^2; \quad \lim_{\gamma \rightarrow \infty} \omega^2 = \infty; \\ \lim_{\gamma \rightarrow 0} [c_p, c_g] &= (a - 1)^{1/2}; \quad \lim_{\gamma \rightarrow \infty} [c_p, c_g] = \infty. \end{aligned}$$

At low frequencies, the behaviour is the same as for shear and Timoshenko. However, both phase and group velocities become unbounded at high frequencies. Such characteristic is of course an anomaly, making possible for very short wavelengths to be predicted immediately at remote locations.

3. Modes

Modes for Euler-Bernoulli are defined as

$$w = e^{j\omega t} \Psi \triangleq e^{j\omega t + rx}$$

with $r \in \mathbb{C}$. Inserting this into (12) gives an fourth order equation in r with solutions

$$r_{\pm}^2 = \frac{(\alpha - 1) \pm [(\alpha - 1)^2 + 4\beta\omega^2]^{1/2}}{2\beta}.$$

Clearly $r_+^2 > 0 \forall \omega^2 \in \mathbb{R}^+$ and $r_+^2 < 0 \forall \omega^2 \in \mathbb{R}^+$ and so the modal shape Ψ can be written as

$$\Psi = d_1 \sin(\lambda_- x) + d_2 \cos(\lambda_- x) + d_3 \sinh(\lambda_+ x) + d_4 \cosh(\lambda_+ x),$$

SS:	$\sin(\lambda_- L) \sinh(\lambda_+ L) = 0$
CL:	$(\lambda_+^2 - \lambda_-^2)/(2\lambda_- \lambda_+) \sin(\lambda_- L) \sinh(\lambda_+ L) - \cos(\lambda_- L) \cosh(\lambda_+ L) + 1 = 0$
FF:	$C_{ff} \sin(\lambda_- L) \sinh(\lambda_+ L) - \cos(\lambda_- L) \cosh(\lambda_+ L) + 1 = 0$
$C_{ff} =$	$\frac{\left[(\lambda_- + \lambda_+) \alpha + \beta \lambda_- \lambda_+ (\lambda_- - \lambda_+) - \lambda_- - \lambda_+ (\lambda_+ - \lambda_- + \alpha (\lambda_- - \lambda_+) - \beta \lambda_- \lambda_+ (\lambda_+ - \lambda_-)) \right]}{\left[2\lambda_- \lambda_+ (\beta \lambda_-^2 + \alpha - 1) (\beta \lambda_+^2 - \alpha + 1) \right]}$

TABLE V. Summary of frequency equations for a prestressed Euler-Bernoulli beam, for the classical simply-supported, clamped and free end conditions.

where $\lambda_+ = r_+$, $\lambda_- = \sqrt{|r_-^2|}$. Inserting this solution into the chosen boundary conditions in table II gives the following equation

$$\mathbf{A} \mathbf{d} = \mathbf{0} \quad (14)$$

where \mathbf{A} is a 4×4 matrix and $\mathbf{d} = {}^T(d_1, d_2, d_3, d_4)$. The associated transcendental frequency equations are given in table V.

III. ANALYSIS

A. Dispersion Curves

Dispersion curves are now plotted for the strings presented in table I. Fig. 2 shows the dispersion relations, phase and group velocities for the low piano string, $D_{\#1}^p$. Note in particular that the phase and group velocities - fig. 2(b), 2(c) - of the Timoshenko and shear models do attain the same limit at large wavenumbers (i.e. $\alpha^{1/2}$), as pointed out in the previous section. The Euler-Bernoulli velocities diverge to infinity. The dispersion curves in fig. 2(a) for Timoshenko and shear attain as expected a slope of 1, whereas the Euler-Bernoulli dispersion relation diverges with a slope of 2. In all figures, however, in the limit of small frequencies the three models show good agreement. The horizontal line in fig. 2(a) represents the limit of human hearing (20kHz): this is the upper limit of interest in musical acoustics. Undoubtedly, the Euler-Bernoulli and shear frequencies are somewhat higher than the Timoshenko frequencies, but the differences seem to be very small. A possible way to quantify the deviation from the Timoshenko model is to define integrated relative "errors" with respect to the Timoshenko model (for the frequencies and group velocities)

in the following way

$$\begin{aligned} \text{dev}_{\omega EB, \omega SH} &= \sup_{[0, \bar{\gamma}]} \left| \frac{\omega_{EB, SH}}{\omega_{TM}} - 1 \right|; \\ \text{dev}_{c_{gEB}, c_{gSH}} &= \sup_{[0, \bar{\gamma}]} \left| \frac{c_{gEB, SH}}{c_{gTM}} - 1 \right|. \end{aligned} \quad (15)$$

where $\bar{\gamma}$ is the wavenumber corresponding the upper limit of human hearing in the Timoshenko model ($\bar{\gamma} \sim 254$ rad/m for $D_{\#1}^p$). Note that the deviation on the phase velocity is necessarily equal to the deviation on the frequencies and therefore is not calculated. With these definitions, one gets

$$\begin{aligned} \text{dev}_{\omega EB} &= 0.016; \quad \text{dev}_{c_{gEB}} = 0.034; \\ \text{dev}_{\omega SH} &= 0.004; \quad \text{dev}_{c_{gSH}} = 0.008. \end{aligned}$$

The deviation of the group velocity is larger than the deviation of the phase velocity by a factor close to 2 and the shear model is more accurate than Euler-Bernoulli by a factor close to 4. Note that the deviation in frequencies, calculated in cents, gives

$$\text{cdev} \triangleq 1200 \log_2 \frac{\omega_{EB, SH}}{\omega_{TM}} = \begin{cases} 27.6 & \text{for EB} \\ 7.2 & \text{for SH} \end{cases}$$

i.e. a little more than a quarter of a semitone for Euler-Bernoulli and a quarter of that for shear. For the $D_{\#1}^p$ string of the double-bass the deviations as defined in Eq.(15) (with $\bar{\gamma} \sim 184$ rad/m) are

$$\begin{aligned} \text{dev}_{\omega EB} &= 0.036; \quad \text{dev}_{c_{gEB}} = 0.073; \\ \text{dev}_{\omega SH} &= 0.008; \quad \text{dev}_{c_{gSH}} = 0.016. \end{aligned}$$

Note in particular that for the piano string $D_{\#1}^p$ both Euler-Bernoulli and shear are closer to Timoshenko than the double-bass string E_1^b : this is in accordance with the fact that the piano string is more slender than the double-bass string (i.e. the ratio between the radius and the

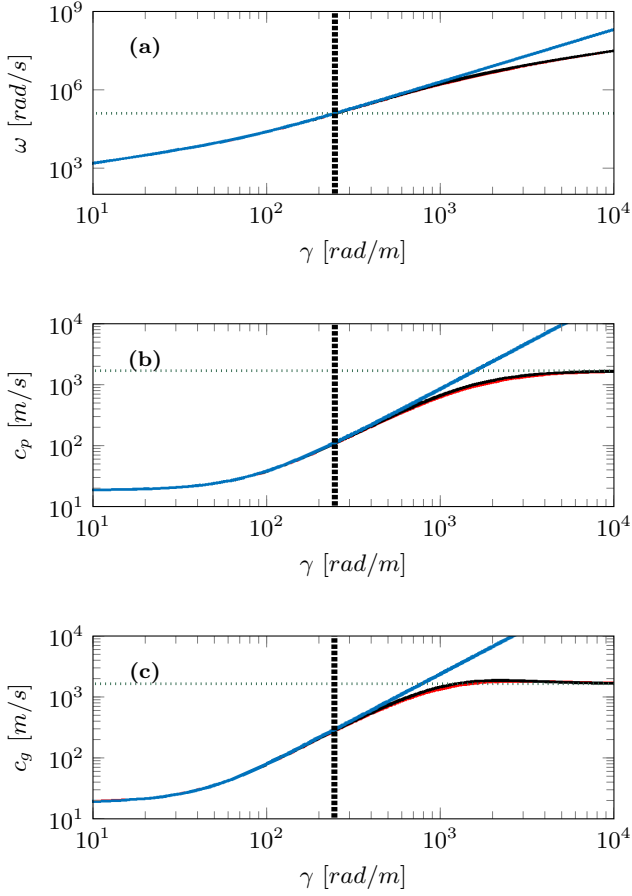


FIG. 2. The piano $D_{\#1}^p$ string. Colour scheme is the following for all figures: blue-EB, black-SH, red-TM. Vertical dashed line is the upper limit of human hearing. (a): dispersion relation; dotted line is upper limit of human hearing. (b): phase velocity; dotted line is the limiting velocity for shear and Timoshenko, $\bar{c}_p = \alpha^{1/2}$; (c): group velocity; dotted line is the limiting velocity for shear and Timoshenko, $\bar{c}_g = \alpha^{1/2}$.

length is smaller for $D_{\#1}^p$ than for E_1^b). Cents deviations are

$$\text{cdev} = \begin{cases} 61.5 & \text{for EB} \\ 14.7 & \text{for SH} \end{cases}$$

In terms of frequencies, the maximum deviations happen at the limit of hearing, and are about half a semitone for Euler-Bernoulli and one sixth of a semitone for shear. For the guitar E_2^g string, with $\bar{\gamma} \sim 255$ rad/m

$$\begin{aligned} \text{dev}_{\omega_{EB}} &= 0.015; \text{dev}_{c_g EB} = 0.031; \\ \text{dev}_{\omega_{SH}} &= 0.004; \text{dev}_{c_g SH} = 0.008. \end{aligned}$$

Cents deviations are

$$\text{cdev} = \begin{cases} 26.1 & \text{for EB} \\ 6.7 & \text{for SH} \end{cases}$$

similar to those for the $D_{\#1}^p$ string.

B. Modal Frequencies

The frequency equations in tables III, IV and V can be solved using an appropriate root finder method, for instance the Newton-Raphson method. Table VI presents a few eigenfrequencies for fixed conditions for the three strings, comparing the three models. Note that, for simply supported conditions and in the case of Euler-Bernoulli, the results in the table are consistent with the well known formula for inharmonicity, i.e.¹⁹

$$f_n = n f_0 \sqrt{1 + n^2 B}; \quad B = \frac{\pi^3 E r^2}{64 L^2 T_0}$$

The frequencies for shear and Timoshenko are very close to the values for Euler-Bernoulli. The modal frequencies for the piano $D_{\#1}^p$ string are plotted in fig. 3, for the three models. It is seen that for low mode numbers the eigenfrequencies are coincident, and they start to depart as the mode number is increased. Fig. 4 is

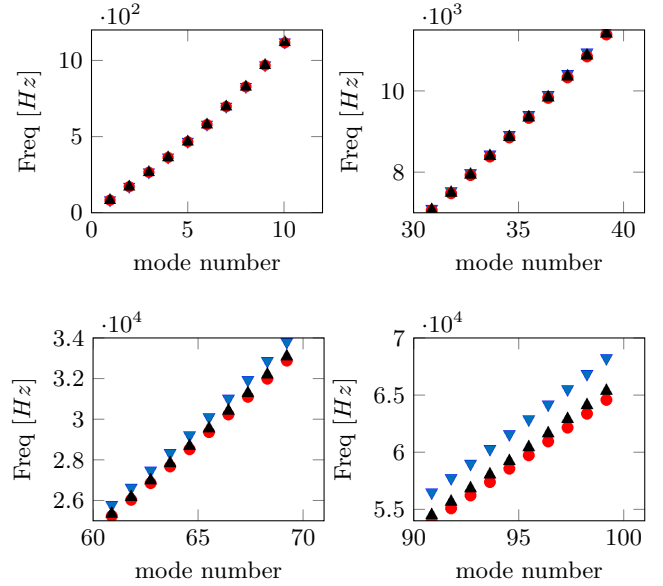


FIG. 3. Eigenfrequencies for the E_2^g string under clamped boundary conditions for Timoshenko (black circles), shear (red triangles), Euler-Bernoulli (blue downward triangles), for different modal numbers.

the deviation in cents of the shear and Euler-Bernoulli models from Timoshenko, over a large frequency range, for simply supported boundary conditions. The three plots correspond to the three strings E_1^b , $D_{\#1}^p$, E_2^g . The limit of human hearing (20kHz) is marked with a vertical dashed line; the two horizontal lines are the 5 and 50 cents boundaries. It is seen that, as expected, E_1^b is the string which shows a larger deviation, and also the string for which deviation sets in at smaller frequencies compared to the other two models. Deviation in cents at the limit of hearing are not dissimilar from those already given for the dispersion curves, in section III.A. Basically, Euler-Bernoulli deviates by about half a semitone at the limit of human hearing, for the bass string, whereas shear deviates by about a quarter of that. In

CLAMPED				S. SUPPORTED			
mode	E_1^b	$D_{\#1}^p$	E_2^g	E_1^b	$D_{\#1}^p$	E_2^g	
1	44.63	39.43	86.36	41.20	38.93	81.85	TM
	44.63	39.43	86.36	41.20	38.93	81.85	SH
	44.63	39.43	86.36	41.20	38.93	81.85	EB
10	684.5	402.1	1109	640.7	396.9	1.055	TM
	684.7	402.1	1110	640.8	397.0	1.055	SH
	685.1	402.1	1110	641.1	397.0	1.055	EB
50 ($\times 10^3$)	12.47	2.782	17.30	12.24	2.750	17.00	TM
	12.54	2.784	17.37	12.31	2.751	17.05	SH
	12.75	2.786	17.55	12.51	2.753	17.22	EB
100 ($\times 10^3$)	46.10	8.727	64.50	45.72	8.656	63.93	TM
	46.91	8.742	65.30	46.53	8.671	64.72	SH
	50.02	8.780	68.08	49.53	8.707	67.40	EB

TABLE VI. Collection of a few eigenfrequencies under clamped and simply supported conditions, comparison of TM,SH,EB over the three strings under study. Values in Hz.

addition, note that, for the bass, the 5 cents boundary is overtaken at about 2 kHz for Euler-Bernoulli and 7 kHz for shear. For the piano and guitar strings, such boundary is overtaken at around 4kHz for Euler-Bernoulli and above 10kHz for shear.

IV. DISCUSSION AND CONCLUSIONS

Three models were assessed describing stiff linear string vibrations. The fundamental model is that due to Timoshenko which can be simplified to yield a number of other models. Here two such simplifications were considered, the shear and the Euler-Bernoulli models. Dispersion properties were given for all models, and plots of dispersion curves showed good agreement of the three models at low frequencies. In fact, for the systems of interest in this work (three thick strings) the largest discrepancies were observed for the case of the double-bass low E, for which the Euler-Bernoulli deviates by about 60 cents from Timoshenko, and shear by about 14 cents. The other cases showed a much better agreement. Modes and modal frequencies were also calculated. Transcendental modal equations were given for all models, for the classic combination of boundary conditions (simply-supported, clamped and free). Plotting the modal frequencies and related deviations showed good agreement with the results obtained by observing the dispersion curves only. In particular, it was noted that the shear model is very close to the Timoshenko model and that deviation larger than 5 cents happen in the higher part of the frequency spectrum. It is concluded that - assuming Timoshenko to be the most exact stiff string vibration

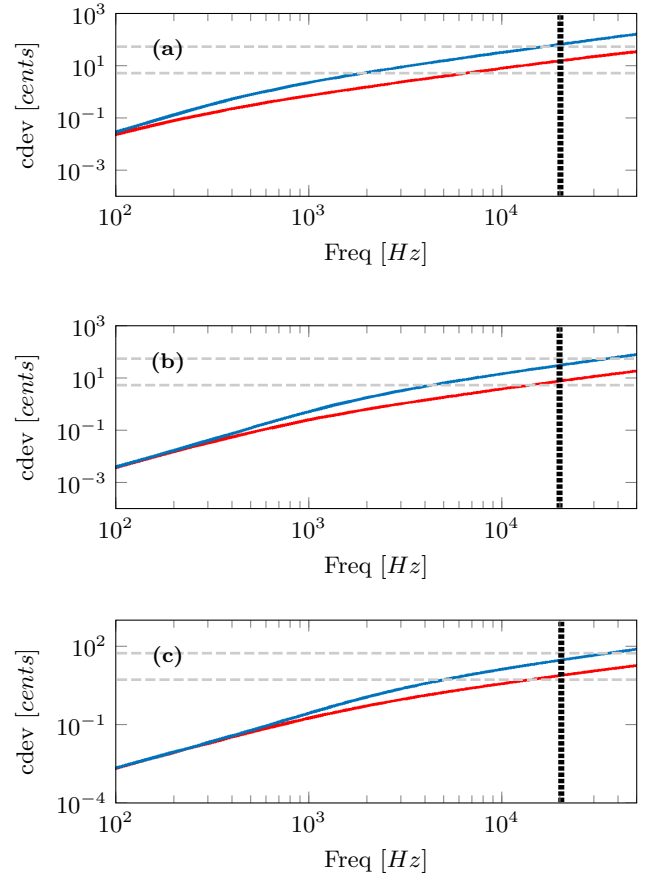


FIG. 4. Cent deviations under simply-supported boundary conditions, for: (a) E_1^b ; (b): $D_{\#1}^p$; (c): E_2^g . Red line: deviation of shear model; blue line: deviation of Euler-Bernoulli model. Vertical dashed line: limit of human hearing (20kHz). Horizontal dashed lines: 5 and 50 cents limits.

model - the Euler-Bernoulli model is an excellent first-approximation. It also concluded that the shear model is a far better approximation and that, to any practical application, it could be used as an easier replacement for the Timoshenko model.

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